

# BOUNDARIES AND POLYHEDRAL BANACH SPACES

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**ABSTRACT.** We show that if  $X$  and  $Y$  are Banach spaces, where  $Y$  is separable and polyhedral, and if  $T : X \rightarrow Y$  is a bounded linear operator such that  $T^*(Y^*)$  contains a boundary  $B$  of  $X$ , then  $X$  is separable and isomorphic to a polyhedral space. Some corollaries of this result are presented.

## 1. THE MAIN RESULT

Let  $X$  be a Banach space. A subset  $B \subset S_{X^*}$  of the unit sphere  $S_{X^*}$  of  $X$  is called a *boundary* of  $X$  if, for any  $x \in X$ , there is  $f \in B$  satisfying  $f(x) = \|x\|$ . From the Krein-Milman Theorem, it follows that the set  $\text{ext } B_{X^*}$  of all extreme points of the unit ball  $B_{X^*}$  of  $X^*$  is a boundary. Easy examples show that a boundary may be a proper subset of  $\text{ext } B_{X^*}$ . A separation theorem shows that, for any boundary  $B$ , we have  $w^*\text{-cl co } B = B_{X^*}$ . It is clear that if  $X$  is infinite-dimensional, then any boundary of  $X$  must be infinite. If  $X$  has a countably infinite boundary then it is separable and isomorphically polyhedral (see [F]).

The following theorem is the main result of this paper.

**Theorem 1.1.** *Let  $X$  and  $Y$  be Banach spaces, where  $Y$  is separable and polyhedral. Assume that  $T : X \rightarrow Y$  is a bounded linear operator, such that  $T^*(Y^*)$  contains a boundary  $B$  of  $X$ . Then  $X$  is separable and isomorphic to a polyhedral space.*

The proof of Theorem 1.1 uses the following result.

**Proposition 1.2.** *Let  $T : X \rightarrow Y$  be a linear bounded operator, such that  $T^*(Y^*)$  contains a boundary  $B$  of  $X$ . Then  $T^*(Y^*)$  is norm dense in  $X^*$ .*

*Proof.* We make use of the so-called (I)-property (see [FL]). Let  $K \subset X^*$  be  $w^*$ -compact and convex, and suppose that  $B \subset K$ . We say that  $B$  has the (I)-property if, whenever  $B = \bigcup_{i=1}^{\infty} B_i$ , where  $B_i \subset B_{i+1}$ , then the set  $\bigcup_{i=1}^{\infty} w^*\text{-cl co } B_i$  is norm-dense in  $K$ . It is

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*Date:* April 7, 2014.

*2010 Mathematics Subject Classification.* 46B20.

*Key words and phrases.* Polyhedral norms, renormings, boundaries, polytopes.

The first named author is supported by Israel Science Foundation, Grant 209/09. The second and third named authors are supported financially by Science Foundation Ireland under Grant Number ‘SFI 11/RFP.1/MTH/3112’. The third named author is also supported by FEDER-MCI MTM2011-22457 and the project of the Institute of Mathematics and Informatics, Bulgarian Academy of Science.

proved in [FL] that, for  $K$  as above, any boundary  $B$  of  $K$  has the (I)-property ( $B \subset K$  is a boundary of  $K$  if, for any  $x \in X$ , there is  $f \in B$  such that  $f(x) = \max x(K)$ ).

Put  $K = B_{X^*}$ , and  $B_i = T^*(iB_{Y^*}) \cap B$  for  $i = 1, 2, \dots$ . Clearly,  $\bigcup_{i=1}^{\infty} w^*\text{-cl co } B_i \subset T^*(Y^*)$ , and the result follows.  $\square$

*Proof of Theorem 1.1.* Without loss of generality, we assume that  $\|T\| = 1$ . Since  $Y$  is a separable polyhedral space, it follows that  $Y^*$  is separable (see [F]), and hence by Proposition 1.2,  $X^*$  and  $X$  are separable too. Next, it is easily seen that  $T$  is injective. Also, without loss of generality, we can assume that  $T$  is dense embedding, i.e.  $\text{cl } T(X) = Y$  (if necessary, pass to the subspace  $Y_1 = \text{cl } T(X)$  and the operator  $T_1 : X \rightarrow Y_1$ ). In particular, we can assume that  $T^*$  is injective. Now set

$$W_n = T^{*-1}(nT^*(B_{Y^*}) \cap B_{X^*}), \quad n = 1, 2, \dots$$

Clearly,

$$B_{Y^*} \subset W_n \subset nB_{Y^*}, \quad n = 1, 2, \dots,$$

(recall that  $\|T\| = 1$ ).

The  $W_n$  are convex centrally symmetric  $w^*$ -compact bodies in  $Y^*$  (with  $Y$  separable and polyhedral). Choose a sequence  $\{\varepsilon_n\}_{n=1}^{\infty}$  of positive numbers tending to 0. By [DFH, Theorem 1.1], each such body  $W_n$  can be approximated by a convex centrally symmetric  $w^*$ -compact body  $A_n$  having a countable boundary, say  $\{\pm h_i^n\}_{i=1}^{\infty}$ , such that

- (a)  $(1 - \varepsilon_n)A_n \subset W_n \subset A_n$ , and
- (b) no  $w^*$ -limit point of  $\{\pm h_i^n\}_{i=1}^{\infty}$  is a support point of  $A_n$ , for any  $y \in Y$ .

Define

$$U^* = w^*\text{-cl co } \bigcup_{n=1}^{\infty} (1 + \varepsilon_n)T^*(A_n), \quad t_i^n = T^*h_i^n, \quad n = 1, 2, \dots, \quad \tilde{B} = \{\pm(1 + \varepsilon_n)t_i^n\}_{i,n=1}^{\infty}.$$

It is easily seen that  $U^*$  is a convex centrally symmetric  $w^*$ -compact set in  $X^*$ , and  $U^* = w^*\text{-cl co } \tilde{B}$ . Moreover,  $B \subset U^*$  and hence  $B_{X^*} \subset U^*$ .

It follows that  $U^*$ , as the unit ball, defines an equivalent dual norm on  $X^*$ . We denote the corresponding norm on  $X$  by  $\|\cdot\|$ , and show that  $(X, \|\cdot\|)$  is polyhedral. To prove this statement, it is enough to check that  $\tilde{B}$  has (\*) (see [FLP]), i.e. no  $w^*$ -limit point of  $\tilde{B}$  is a support point of  $U^*$ , for any  $x \in X$ .

If  $f$  is a  $w^*$ -limit point of  $\tilde{B}$  then, by the separability of  $X$ , we may find a sequence of distinct points in  $\tilde{B}$  that converges to  $f$  in the  $w^*$ -topology. There are two possibilities. The first possibility is that, for some fixed  $n$ , we can write  $f = w^*\text{-lim}_k (1 + \varepsilon_n)t_{i_k}^n \in (1 + \varepsilon_n)T^*(A_n)$ , where  $i_1 < i_2 < \dots$ . Then  $T^{*-1}f$  is a  $w^*$ -limit point of the set  $\{\pm(1 + \varepsilon_n)h_i^n\}_{i=1}^{\infty}$  and, by (b) above, it cannot be a support point of  $(1 + \varepsilon_n)A_n$  for any  $y \in Y$ . Therefore,  $f$  is not a support point of  $(1 + \varepsilon_n)T^*(A_n)$ , for any  $x \in X$ . Since  $f \in (1 + \varepsilon_n)T^*(A_n) \subset U^*$  it follows that  $f$  is not a support point of  $U^*$ , for any  $x \in X$ .

If the first possibility above does not hold then we can write  $f = w^*\text{-}\lim_k (1 + \varepsilon_{n_k})t_{i_k}^{n_k}$ , where  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Assume to the contrary that  $f$  is a support point of  $U^*$  with  $f(x_0) = \max x_0(U^*) \neq 0$ ,  $x_0 \in X$ . By (a), it is easily seen that  $f \in B_{X^*}$ . Since  $B_{X^*} \subset U^*$ , it follows that  $f(x_0) = \max x_0(B_{X^*}) = \|x_0\| \neq 0$ . As  $B$  is a boundary of  $X$ , there is  $g \in B$  satisfying  $g(x_0) = \|x_0\|$ . Next, because  $B \subset T^*(Y^*)$ , it easily follows that  $B \subset \bigcup_{n=1}^{\infty} T^*(A_n)$ , and hence  $g \in T^*(A_m)$  for some  $m$ . However,  $(1 + \varepsilon_m)g \in U^*$  implies  $\max x_0(U^*) \geq (1 + \varepsilon_m)\|x_0\|$ , contradicting  $\|x_0\| = f(x_0) = \max x_0(U^*) \neq 0$ . The proof is complete.  $\square$

## 2. CONSEQUENCES

**Corollary 2.1.** *Let  $X$  be a Banach space and  $Y_i$ ,  $i = 1, 2, \dots$ , be a sequence of separable isomorphically polyhedral Banach spaces. Let  $B$  be a boundary of  $X$  and  $T_i : X \rightarrow Y_i$  a sequence of bounded linear operators, such that*

$$B \subset \bigcup_{i=1}^{\infty} T_i^*(Y_i^*).$$

*Then  $X$  is separable and isomorphically polyhedral.*

*Proof.* Without loss of generality we can assume that  $Y_i$  is polyhedral and  $\|T_i\| = 1$  for all  $i$ . Let  $Y = (\bigoplus_{i=1}^{\infty} Y_i)_{c_0}$ . Clearly  $Y$  is a separable isomorphically polyhedral space. Define  $T : X \rightarrow Y$  by  $(Tx)_i = i^{-1}T_i x \in Y_i$ . Evidently,  $Y^* = (\bigoplus_{i=1}^{\infty} Y_i^*)_{\ell_1}$  and if  $f \in Y^*$  and  $x \in X$ , we have

$$(T^*f)(x) = f(Tx) = \sum_{i=1}^{\infty} i^{-1}f_i(T_i x) = \sum_{i=1}^{\infty} (i^{-1}T_i^* f)(x),$$

where  $f = (f_i)_{i=1}^{\infty}$  and  $\|f\| = \sum_{i=1}^{\infty} \|f_i\|$ . In particular, if  $g \in Y_i^* \subset Y^*$  we have  $T^*g = i^{-1}T_i^*g$ , whence

$$B \subset \bigcup_{i=1}^{\infty} T_i^*(Y_i^*) \subset T^*(Y^*).$$

Apply Theorem 1.1 to finish the proof.  $\square$

**Corollary 2.2.** *Assume that  $X$  has a boundary  $B$  which is contained in a set of the form*

$$\bigcup_{i=1}^{\infty} w^*\text{-cl co } K_i,$$

*where the  $K_i$  are countable  $w^*$ -compact subsets of  $X^*$ . Then  $X$  is isomorphically polyhedral.*

*Proof.* Let  $Y_i$  be the isomorphically polyhedral space  $C(K_i)$ , and define  $T_i : X \rightarrow Y_i$  by  $(T_i x)(t) = t(x)$ ,  $x \in X$ ,  $t \in K_i$ . Then  $w^*\text{-cl co } K_i \subset T^*(B_{C(K_i)^*})$ , and we can apply Corollary 2.1.  $\square$

We will see later in Remark 2.6 that we cannot drop the requirement that the  $K_i$  above are  $w^*$ -compact. Our final result includes an application of the material above, stated in terms of M-bases.

**Definition 2.3.** Let  $\{x_i\} \subset X$  be an M-basis of a Banach space  $X$ , having biorthogonal sequence  $\{x_i^*\} \subset X^*$ . We call a subset  $A \subset X^*$  *summable* if  $\sum_{i=1}^{\infty} |f(x_i)| < \infty$  for all  $f \in A$ .

**Lemma 2.4.** Let  $B = \{\pm f_i\} \subset S_{X^*}$  be a countable boundary of  $X$ , take a sequence of numbers  $\{\varepsilon_i\}$ ,  $0 < \varepsilon_i < \frac{1}{2}$ ,  $\lim_i \varepsilon_i = 0$ , and a sequence of vectors  $\{t_i\} \subset X^*$  satisfying  $\|f_i - t_i\| < \varepsilon_i$ . Then the sequence  $\pm h_i = \pm(1 + 2\varepsilon_i)t_i$ ,  $i = 1, 2, \dots$ , is a boundary having (\*) with respect to norm it generates, given by

$$|||x||| = \sup_i |h_i(x)|, \quad x \in X.$$

*Proof.* First, note that for any  $x \in X$ ,  $x \neq 0$ , we have

$$|||x||| > \|x\|. \quad (1)$$

Indeed, if  $f_i(x) = \|x\|$  then

$$\begin{aligned} |||x||| &\geq (1 + 2\varepsilon_i)t_i(x) \\ &\geq (1 + 2\varepsilon_i)f_i(x) - (1 + 2\varepsilon_i)\|t_i - f_i\| \|x\| \\ &\geq (1 + 2\varepsilon_i)(1 - \varepsilon_i)\|x\| > \|x\|. \end{aligned}$$

Put  $V^* = \{f \in X^* : |||f||| \leq 1\}$ ,  $S_{V^*} = \partial V^*$ . By using the Hahn-Banach Theorem and (1), we easily obtain  $B_{X^*} \subset V^*$  and, again by using (1), we see that no functional  $f \in S_{V^*} \cap B_{X^*}$  (if any) attains its norm with respect to  $||| \cdot |||$ . However, any  $w^*$ -limit point  $g$  of  $B_1$  satisfying  $|||g||| = 1$  (if any) lies in  $B_{X^*}$  (recall that  $\lim_i \varepsilon_i = 0$ ). The proof is complete.  $\square$

**Theorem 2.5.** For a separable Banach space  $X$ , the following assertions are equivalent.

- (a)  $X$  admits a boundary  $B$  and a bounded linear operator  $T : X \rightarrow c_0$ , such that  $T^*(c_0^*) \supset B$ .
- (b)  $X$  admits a boundary  $B$  and a bounded linear operator  $T : X \rightarrow Y$  into a polyhedral space  $Y$ , such that  $T^*(Y^*) \supset B$ .
- (c)  $X$  is isomorphically polyhedral.
- (d)  $X$  admits an equivalent norm having a boundary  $B$ , which is summable with respect to a normalized M-basis  $\{x_i\}$  with bounded biorthogonal sequence  $\{x_i^*\}$ .

*Proof.* (a)  $\Rightarrow$  (b) is trivial, while (b)  $\Rightarrow$  (c) is Theorem 1.1. To prove (c)  $\Rightarrow$  (d), we can assume without loss of generality that  $X$  polyhedral. By [F1],  $X$  admits a countable boundary  $\{f_i\}$ , and  $X^*$  is separable. It is well-known (see for instance [FHHMVZ, Theorem 4.59]), that  $X$  admits a (shrinking) normalized M-basis  $\{x_i\}$  with bounded biorthogonal sequence  $\{x_i^*\}$ . By using Lemma 2.4, we easily obtain a sequence  $B = \{h_i\} \subset \text{span}\{x_i^*\}$

which is a boundary of  $X$  with respect to an equivalent norm. Clearly,  $B$  is summable with respect to the M-basis  $\{x_i\}$ . Finally, we prove (d)  $\Rightarrow$  (a). Let  $B$  be a boundary of  $X$  which is summable with respect to a normalized M-basis  $\{x_i\}$  with bounded biorthogonal sequence  $\{x_i^*\}$ . Define  $T : X \rightarrow c_0$  by

$$Tx = (x_i^*(x))_{i=1}^\infty, \quad x \in X.$$

Evidently,  $\|T\| = \sup_i \|x_i^*\| < \infty$ . If  $\{e_i\}$  is the natural basis of  $\ell_1 = c_0^*$ , then it is easily seen that  $T^*e_i = x_i^*$ ,  $i = 1, 2, \dots$ . Since  $B$  is summable, it follows that  $T^*(c_0^*) \supset B$ .  $\square$

**Remark 2.6.** We cannot replace the  $x_i^*$  in Theorem 2.5 (d) even with a normalized basis of  $X^*$ , if its biorthogonal sequence does not belong to  $X$ . Indeed, if  $X^*$  is isomorphic to  $\ell_1$  then it admits a normalized basis, with respect to which every element of  $X^*$  is an absolutely summable combination. However, there exist non-isomorphically polyhedral Banach spaces having duals isomorphic to  $\ell_1$ . The spaces in e.g. [BD] and [AH] have duals isomorphic to  $\ell_1$  but do not contain any isomorphic copies of  $c_0$ , and in order for a Banach space to be isomorphically polyhedral, it is necessarily  $c_0$ -saturated [F]. The same examples show that it is necessary for the  $K_i$  in Corollary 2.2 to be  $w^*$ -compact.

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